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THE EXISTENCE AND UNIQUENESS OF SELF-SIMILAR SOLUTIONS INVOLVING JOUGUET POINTS ON THE SHOCK ADIABATIC CURVE†

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The self-similar solutions of one-dimensional unsteady equations of the hyperbolic type which express the conservation laws are considered. A number of special features have been revealed [1–4] in the non-linear theory of elasticity: the absence of uniqueness, the absence of a continuous dependence of a solution on the parameters and rapid changeovers of a wave of one type with the emission of a wave of another type. It has been shown that all of these phenomena have a common character and are associated with the presence of a "foreign" Jouguet point on the shock adiabat. The velocity of a shock wave of a certain type at this point is equal to the characteristic velocity downstream of a shock wave of another neighbouring and slower type. Hence, in the case of hyperbolic systems, which express the conservation laws, the possibility exists of judging the most important properties of self-similar solutions as a whole using certain properties of the shock adiabats.

1. PRELIMINARY INFORMATION

We shall consider hyperbolic systems of equations which express the conservation laws [5] in the case of two independent variables x and t

$$\partial f_i(u_k)/\partial t+\partial g_i(u_k)/\partial x=0,\quad [g_i]-W[f_i]=0;\ i,k=1,2,\dots,n \tag{1.1}$$

Here, $u_k(x, t)$ are unknown functions, and f_i and g_i are specified functions of u_k which give the system of equations a specified form. It is assumed that the differential equations describe smooth solutions and the finite equations represent a complete system of relations at the shock wave, where W is the shock-wave velocity, $[g_i] = g_i(u_k^+) - g_i(u_k^-)$ is the "jump" in the function g_i and u_k^- refers to the state upstream of the shock wave while u_k^+ refers to the state downstream.

The conditions for the evolutionary character of shock waves of the kth type ($k = 1, 2, \dots, n$) have the form [5–6]

$$c_k^-\leqslant W\leqslant c_{k+1}^-,\quad c_{k-1}^+\leqslant W\leqslant c_k^+ \tag{1.2}$$

where c_j ($j = 1, 2, \dots n$) are the velocities of the characteristics of system (1.1) and one puts $c_0^+ = -\infty$, $c_{n+1}^- = \infty$. On the shock adiabatic curve, the equation of which, in the general case, can be obtained by solving relations at the shock wave (1.1) for specified u_k^- in the form $u_i^+ = u_i^+(W)$, conditions (1.2) separate into "evolutionary segments". The "Jouguet conditions", which express the equality of the shock-wave velocity W to any characteristic velocity downstream or upstream of the shock wave (in the latter case, we shall speak of an "upstream" Jouguet condition), are satisfied at the ends of the segments.

Inequalities (1.2) and the shock adiabat are often represented on an evolution diagram, where the characteristic velocities c_k^- and the current value of W on the shock adiabat are plotted along the abscissa on a certain scale and the quantities c_i^+ and W are plotted along the ordinate without preserving the scales but preserving the continuity of the change in W in the shock adiabat and the signs of inequalities (1.2).

Such a diagram for the quasitransverse waves in an elastic medium with a small anisotropy [7, 8] is shown in Fig. 1 as an example. The evolutionary segments of the shock adiabat lie in the hatched "evolutionary" rectangles. In the space u_i, points A correspond to the initial point of the shock adiabat $u_i = u_i^-$ at which the n branches of the shock adiabat intersect one another. These branches turn out to be spaced in the evolution diagram on account of the fact that a point A is shown there in the form of n points with the coordinates $W = c_k^-$, $W = c_k^+$, $k = 1, 2, \dots, n$. The behaviour of the shock adiabat in

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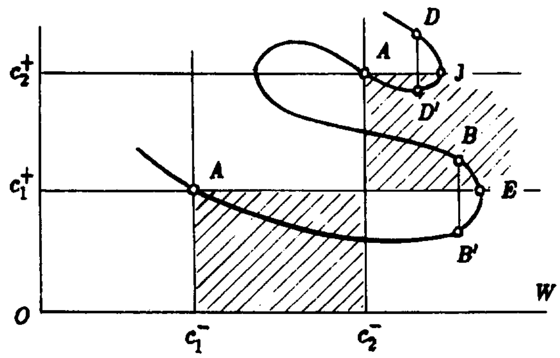

Fig. 1.

the diagram in the neighbourhood of each of the points A follows from the equality obtained in [5] for small discontinuities

$$W = (c_k^- + c_k^+)/2 \tag{1.3}$$

and, moreover, the order of magnitude of the error in (1.3) is no greater than the square of the jump in the magnitudes of the quantities at the shock wave. It follows from (1.3) that the shock adiabat has a negative slope at all points A in the evolution diagram. Hence, in the diagram and, also, in the space u_i, only a single evolutionary segment borders upon the initial point for each branch of the shock adiabat. In the opposite direction, the integral curve (IC) of the Riemann wave, for which the characteristic velocity decreases as it departs from the initial point [5], touches it, preserving the tangent and the curvature vector.

As was shown in [9, 10], the existence of an extremum of W at this point follows from the satisfaction of the Jouguet condition $W = c_j^+$ at a certain point on the shock adiabat and the converse also holds. Furthermore, it was shown that the shock adiabat in the space u_i touches the IC of the Riemann wave at the Jouguet point corresponding to c_j.

2. SOLUTIONS OF THE SELF-SIMILAR PROBLEM WITH A SHOCK WAVE CLOSE TO "ITS" JOUGUET POINT

We shall consider the self-similar problem of the solution in the form of a system of waves which depend on $\xi = x/t$. The state $u_i = U_i$ "upstream" of the system of waves will be assumed to be known for large ξ and specified when investigating the solution for state $u_i = u_i^*$ "downstream" of the system of waves for large negative values of ξ. Naturally, the parameters of the problem which are specified and determined can be easily interchanged assuming that the values of u_i are given when $\xi = -\infty$ and we determine the values of u_i when $\xi = \infty$.

The solution of the self-similar problem may consist of discontinuities located in the x, t plane on the lines $x/t =$ const and self-similar, or centred, Riemann waves in which the characteristic velocity decreases as ξ decreases, which fixes the direction of the motion along the IC.

We shall initially consider the solution of the self-similar problem when one of the discontinuities is close to "its" Jouguet point which, in the evolution diagram, corresponds to the intersection of the shock adiabat with the upper boundary of an evolutionary rectangle (point J in Fig. 1 is such a point). It has already been mentioned that, in the space u_i, an evolutionary segment of the shock adiabat which corresponds to discontinuities of the kth type touches the IC of the Riemann wave at a Jouguet point corresponding to c_k.

In the case of a general position, a part of the IC of the Riemann wave serves as a continuation of an evolutionary segment of the shock adiabat located close to its non-evolutionary part, in which the characteristic velocity decreases as it departs from the Jouguet point.

Actually, two points which may be as close as desired, lie on different sides of the Jouguet point and correspond to one and the same velocity W (these are the points D and D' in Fig. 1) are found by virtue of the extremal nature of W at the Jouguet point. It is obvious that these points can represent the initial and final points of a certain discontinuity which propagates at a velocity W and for which the relations on shock wave (1.1) are satisfied.

It was mentioned in Section 1 that $c_k^- < W < c_k^+$ in a small evolutionary shock wave. In the evolution diagram, this corresponds to a downwards jump across the horizontal line, which corresponds to $W = c_k^+$. In the case being considered, points D and D' can correspond to the state upstream and downstream of an evolutionary shock wave, respectively. In the space u_i as well as in the evolution diagram, there is a jump from a non-evolutionary segment of the shock adiabat to an evolutionary segment. If one draws the IC of the Riemann wave through point D', then, according to the results in [5] which are mentioned in Section 1, it will pass by point D at a distance of the order of ε^3, where ε is the distance between points D and D' in the space of u_i. In this case, the characteristic velocity decreases on the IC in the direction from D to D'. As point D approaches the Jouguet point J, we obtain confirmation of what was stated earlier that, at the Jouguet point, an evolutionary segment of a shock adiabat is smoothly extend beyond "its" Jouguet point of the Riemann wave IC at which the characteristic velocity decreases. In fact, this part of the IC is used to construct self-similar solutions and will subsequently be referred to as the integral curve of a self-similar Riemann wave.

The mutual arrangement of the above-mentioned curves does not prevent a continuous dependence of the solution of the self-similar problem in the neighbourhood of a Jouguet point of its own type (such as point J in Fig. 1) on the parameters which specify the state downstream of the self-similar system of waves which propagate in accordance with the state which is specified upstream. The above-mentioned continuous dependence is obvious in the case when the shock wave with the Jouguet point being considered is the fastest in the system of waves giving the solution of the problem under consideration. In the general case, when there is a small change in the state downstream of the system of waves, the solution will contain either an evolutionary shock wave of the fast type being considered close to the Jouguet point or a fast Jouguet shock wave with a fast self-similar Riemann wave following it. If the changed state downstream of the system of waves does not lie in an evolutionary segment of the shock adiabat or in the part of the IC of the Riemann wave which extends it, this leads to the appearance of other waves (different from fast waves) of small amplitude. In this case, the problem always turns out to be solvable since the system of vectors, which are tangents to the curves specifying the change in the quantities in these waves, and the tangent to the shock adiabatic curve at the Jouguet point form a non-degenerate system of vectors representing the complete system of eigenvectors corresponding to small perturbations relative to the state defined by the Jouguet point.

In the case when the Jouguet point being considered does not correspond to the fastest wave, it is more difficult to investigate the solvability of the problem of determining the amplitudes of the waves. However, the solvability of this has been proved in problems of elastic waves in weakly anisotropic media [1], and, generally, the solvability of the problem of determining the amplitudes of the waves corresponds to the general situation.

3. SOLUTIONS OF THE SELF-SIMILAR PROBLEM WHEN THERE IS A "FOREIGN" JOUGUET POINT ON THE SHOCK ADIABAT

In the case when there is a "foreign" Jouguet point, that is, when the equality $W = c_{k-1}^+$ is satisfied for a shock wave of the kth type, the shock adiabat leaves an evolutionary rectangle in the diagram, intersecting its lower boundary (point E is such a Jouguet point in Fig. 1). In this case, in the space of the variables u_i, the segment of the shock adiabat which corresponds to the kth shock waves is in contact with the IC of the $(k-1)$th Riemann wave at the Jouguet point (which corresponds to c_{k-1}). By arguing in the same way as in Section 2, we obtain that the characteristic velocity c_{k-1} on the non-evolutionary extension of the shock adiabat increases as the distance from the Jouguet point increases. This means that, in the space of u_i, the evolutionary part of the shock adiabat which corresponds to the kth shock waves together with the self-similar part of the IC of the $(k-1)$th Riemann wave (for which the characteristic velocity decreases as the distance from the Jouguet point increases and the wave itself expands as t increases) has a cuspidal point at the Jouguet point (see Fig. 2, where the corresponding parts of the shock adiabat and the self-similar Riemann wave are indicated by the letters S_k and R_{k-1}, and the non-evolutionary segment of the shock adiabat is indicated by the dashed line).

If, in a small neighbourhood of the Jouguet point E in the space of u_i, the shock adiabat S_k and the Riemann wave R_{k-1} are replaced by their common tangent, it is obvious that a change in the parameters which characterize the two above-mentioned waves leads to a motion along one and the same line. It follows that, when the amplitudes of the n different waves, which are available when constructing the solution, are varied, the point u_i^* transverses an $(n-1)$-dimensional hypersurface rather than the whole of the n-dimensional neighbourhood of the Jouguet point. This demonstrates the non-existence of a solution in the accepted "linear" approximation.

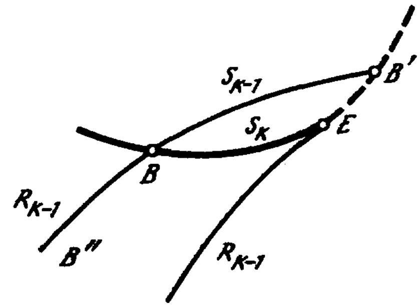

Fig. 2.

For a more accurate construction of the close self-similar solutions, we note that each point (B in Fig. 2, for example), which belongs to an evolutionary segment S_k of the shock adiabat, corresponds, by virtue of its position, to the state downstream of kth shock wave beyond which it can follow either the $(k-1)$th evolutionary shock wave S_{k-1} or the continuous, self-similar $(k-1)$th Riemann wave R_{k-1}. In the space of u_i, the state downstream of the above-mentioned waves belongs to a curve which is composed of the evolutionary part BB' of the shock adiabat S_{k-1} and the curve BB'' of the Riemann wave R_{k-1} which serves as an extension of its integral curve. As point B moves along S_k, the composite curve $B''BB'$ sweeps over a certain two-dimensional surface, for the points of which it is possible to construct a solution which consists of a sequence of the kth evolutionary shock wave and the $(k-1)$th wave (a continuous wave or a shock wave) and is close to the solution which corresponds to the Jouguet point E.

We shall show that this is a surface with a boundary on the other side of which a self-similar solution of the type which has been described does not exist. We shall now find the curve which represents this boundary.

If a sufficiently small evolutionary shock wave of the $(k-1)$th type follows at a distance B an evolutionary shock wave of the kth type moving at a velocity W_B, then its velocity is less than W_B since $W_B - c^+_{k-1} > 0$ downstream of a shock wave of the kth type, according to (1.2). If, while keeping B constant, the magnitude of the second jump is increased, its velocity will increase and, when the point representing the state downstream of this shock wave arrives at position B' (Fig. 1), the velocity of this $(k-1)$th shock-wave becomes equal to W_B. In the physical space, the shock waves of the kth and $(k-1)$th types will merge, forming a single non-evolutionary shock wave corresponding to point B'. A further increase in the magnitude of a shock wave of the $(k-1)$th type has no physical meaning and does not correspond to solutions of a self-similar problem since a $(k-1)$th shock wave cannot overtake a kth type shock wave. This leads to the fact that, in the space of the variables u_i, the domain where a solution exists, consisting of two shock waves of the kth and $(k-1)$th types, S_kS_{k-1}, represents a certain surface bounded by a curve consisting of points B and B', that is, the shock adiabat. On the other side of the shock adiabat, as was stated above, a self-similar solution can be constructed, consisting of a shock wave of the kth type, S_k, and a Riemann wave of the $(k-1)$th type, R_{k-1}, which we shall denote by S_kR_{k-1}. The two-dimensional surface, corresponding to the solutions S_kR_{k-1} and S_kR_{k-1} is bounded by a curve which consists of the non-evolutionary part of the shock adiabat (EB' in Fig. 2) and the IC of the Riemann wave (EC) which touches the shock adiabatic curve at point E.

A self-similar solution cannot exist on the other side of the above-mentioned boundary of the two-dimensional surface and, if it exists, it must have a different structure. As the treatment of waves in an anisotropic elastic body shows [1, 2] and, as follows from general considerations, two cases are possible.

1. A case is possible when a domain in which another solution exists is bounded by the same boundary so that, in the neighbourhood of point E, a solution exists and is unique. For this to be so, it is necessary that the second solution, as well as the first solution which has been considered above, should be genetically associated with the non-evolutionary part of the shock adiabat (EB' in Fig. 2). This is possible if there is a second combination of two shock waves, which is different from that considered above, that merge into a single shock wave on approaching the line representing a non-evolutionary segment of the shock adiabat. Actually, such a situation occurs in the theory of elasticity when $W_J > W_E$ [1, 4]. In this case, at velocities slightly less than W_E, there are two different fast quasitransverse shock waves corresponding to an upper evolutionary rectangle (Fig. 1) which, together with slow waves, can form two combinations corresponding to a non-evolutionary shock wave close to the shock wave corresponding to point E [1, 2, 11].

Note that, although the self-similar solutions obtained as a result of a different decomposition of a non-evolutionary shock wave are not close to one another if their closeness is estimated using the maximum of the modulus of the difference between the solutions, they nevertheless remain close to one another if other measures of their difference which allow for the size of the domain, where these solutions strongly differ, are used.

2. We will now consider the case when the non-evolutionary part of the shock adiabat which is adjacent to a Jouguet point is such that only a single combination (considered above) of the two waves exists and the merging of these two waves corresponds to non-evolutionary shock waves. In the theory of elasticity [1, 4], this holds in the case when $W_J > W_E$, which is shown in Fig. 1. Then, if the self-similar solution holds for all points in the neighbourhood of point E, then the second solution, not being directly associated with the shock adiabatic curve in the neighbourhood of point E, is not close, in the general case, to the solution considered above and the boundary of the domain where this solution holds does not pass, in the general case, close to point E. This means that, in half the neighbourhood of point E, there are two self-similar solutions which are not close to one another. If the parameters defining the first of these solutions are changed, it is possible to intersect the boundary of existence of this solution after which a solution of the type being considered ceases to exist and must instantaneously decompose into another self-similar system of waves. These phenomena have previously been revealed in the investigation of elastic waves [3]. It is obvious from what has been described that these phenomena are associated with certain singularities in the structure of the shock adiabat which can occur in various physical situations.

All the assertions which have been proved for self-similar solutions with an argument x/t are also true in the case of steady self-similar solutions with an argument $\varphi = \operatorname{arctg} y/x$. In this case, the shock polar is mapped onto the evolution diagram. The magnitude of φ will reach an extremum value at the Jouguet points. Points of the shock polar, corresponding to one and the same value of φ, can represent states on different sides of a shock wave, the position of which is determined by this value of φ. As in the preceding discussion, there will be no continuous dependence of the self-similar solution on the parameters if a "foreign" Jouguet point exists on the shock polar. It is obvious that, in this case, a foreign Jouguet point will also be present on the shock adiabat with the corresponding form of the chosen initial conditions. The phenomena discussed in this paper are therefore the absence of a continuous dependence of the solution on the parameters and the fact that the self-similar solutions with an argument $\varphi = \operatorname{arctg} y/x$ may only be non-unique in those media in which these phenomena occur in the case of self-similar solutions with an argument x/t.

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